

Relaxed sector condition

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Abstract

In this note we present a new sufficient condition which guarantees martingale approximation and central limit theorem *a la Kipnis – Varadhan* to hold for additive functionals of Markov processes. This new condition, dubbed the *relaxed sector condition* generalizes several other well-known sector conditions like the (strong) sector condition and the graded sector condition, while also being interesting in its own right.

1 Introduction

The theory of central limit theorems for additive functionals of ergodic Markov processes via martingale approximation was initiated in the mid-1980-s with applications to tagged particle diffusion in stochastic interacting particle systems and various models of random walks in random environment.

The Markov process is usually assumed to be a stationary and ergodic regime. We shall stick to these assumptions in the present note, too. There are however also other type of related results, see e.g. [7], [1], which use different techniques.

In their celebrated 1986 paper [4], C. Kipnis and S. R. S. Varadhan proved a central limit theorem for the reversible case with no assumptions other than the strictly necessary ones. For an early non-reversible extension see [11] where the martingale approximation was applied to a particular model of random walk in random environment.

The theory have since been widely extended by Varadhan and collaborators to include processes with a varying degree of non-reversibility. For a detailed account of these so-called *sector conditions* and the different models they are applied to, see the surveys [8], [5] and the more recent result [3].

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In the present note, we introduce a new sector condition dubbed the *relaxed sector condition*. Apart from appearing to be interesting in its own right, it also provides a new version of the *graded sector condition* with a less technical and more transparent proof.

2 Setup, abstract considerations

We recall the non-reversible version of the abstract Kipnis–Varadhan CLT for additive functionals of ergodic Markov processes, see [4] and [11].

Let $(\Omega, \mathcal{F}, \pi)$ be a probability space: the state space of a *stationary and ergodic* Markov process $t \mapsto \eta(t)$. We put ourselves in the Hilbert space $\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$. Denote the *infinitesimal generator* of the semigroup of the process by G , which is a well-defined (possibly unbounded) closed linear operator on \mathcal{H} . The adjoint generator G^* is the infinitesimal generator of the semigroup of the reversed (also stationary and ergodic) process $\eta^*(t) = \eta(-t)$. It is assumed that G and G^* have a *common core of definition* $\mathcal{C}_0 \subseteq \mathcal{H}$. Let $f \in \mathcal{H}$, such that $(f, \mathbb{1}) = \int_{\Omega} f \, d\pi = 0$. We ask about CLT/invariance principle for

$$N^{-1/2} \int_0^{Nt} f(\eta(s)) \, ds \quad (1)$$

as $N \rightarrow \infty$.

We denote the *symmetric* and *antisymmetric* parts of the generators G, G^* , by

$$S := -\frac{1}{2}(G + G^*), \quad A := \frac{1}{2}(G - G^*).$$

These operators are also extended from \mathcal{C}_0 by graph closure and it is assumed that they are well-defined self-adjoint, respectively, skew self-adjoint operators:

$$S^* = S \geq 0, \quad A^* = -A.$$

Note that $-S$ is itself the infinitesimal generator of a Markovian semigroup on $\mathcal{L}^2(\Omega, \pi)$, for which the probability measure π is reversible (not just stationary). We assume that $-S$ is itself ergodic:

$$\text{Ker}(S) = \{c\mathbb{1} : c \in \mathbb{C}\}.$$

We denote by $R_{\lambda} \in \mathcal{B}(\mathcal{H})$ the resolvent of the semigroup $s \mapsto e^{sG}$:

$$R_{\lambda} := \int_0^{\infty} e^{-\lambda s} e^{sG} \, ds = (\lambda I - G)^{-1}, \quad \lambda > 0,$$

and given $f \in \mathcal{H}$ as above, we will use the notation

$$u_{\lambda} := R_{\lambda} f.$$

The following theorem is direct extension to non-reversible setup of the Kipnis–Varadhan Theorem from [4]. It yields the efficient martingale approximation of the additive functional (1). To the best of our knowledge this non-reversible extension appears first in [11].

Theorem 1 (KV). *With the notation and assumptions as before, if the following two limits hold in \mathcal{H} :*

$$\lim_{\lambda \rightarrow 0} \lambda^{1/2} u_\lambda = 0, \quad (2)$$

$$\lim_{\lambda \rightarrow 0} S^{1/2} u_\lambda =: v \in \mathcal{H}, \quad (3)$$

then

$$\sigma^2 := 2 \lim_{\lambda \rightarrow 0} (u_\lambda, f) \in [0, \infty),$$

exists, and there also exists a zero mean, \mathcal{L}^2 -martingale $M(t)$ adapted to the filtration of the Markov process $\eta(t)$ with stationary and ergodic increments and variance

$$\mathbf{E} (M(t)^2) = \sigma^2 t$$

such that

$$\lim_{N \rightarrow \infty} N^{-1} \mathbf{E} \left(\left(\int_0^N f(\eta(s)) ds - M(N) \right)^2 \right) = 0.$$

In particular, if $\sigma > 0$, then the finite dimensional marginal distributions of the rescaled process $t \mapsto \sigma^{-1} N^{-1/2} \int_0^{Nt} f(\eta(s)) ds$ converge to those of a standard 1d Brownian motion.

Remarks. (1) For the historical record it should be mentioned that the idea of martingale approximation and an early variant of this theorem under the much more restrictive condition

$$f \in \text{Ran}(G),$$

appears in [2]. For more exhaustive historical account and bibliography of the problem see the recent monograph [5].

(2) Conditions (2) and (3) of the theorem are jointly equivalent to the following

$$\lim_{\lambda, \lambda' \rightarrow 0} (\lambda + \lambda') (u_\lambda, u_{\lambda'}) = 0. \quad (4)$$

Indeed, straightforward computations yield:

$$(\lambda + \lambda') (u_\lambda, u_{\lambda'}) = \| S^{1/2} (u_\lambda - u_{\lambda'}) \|^2 + \lambda \| u_\lambda \|^2 + \lambda' \| u_{\lambda'} \|^2.$$

(3) The theorem is a generalization to non-reversible setup of the Kipnis–Varadhan theorem, [4]. The non-reversible formulation appears – in discrete-time Markov chain, rather than continuous-time Markov process setup and with condition (4) – in [11] where it was applied, with bare hand computations, to obtain CLT for a particular random walk in random environment. Its proof follows the original proof of the Kipnis–Varadhan theorem with the difference that spectral calculus is to be replaced by resolvent calculus.

- (4) In continuous-time Markov process setup, it was formulated in [12] and applied to tagged particle motion in non-reversible zero mean exclusion processes. In this paper, the *(strong) sector condition* was formulated, which, together with an H_{-1} -bound on the function $f \in \mathcal{H}$, provide sufficient condition for (2) and (3) of Theorem KV to hold.
- (5) In [10], the so-called *graded sector condition* is formulated and Theorem KV is applied to tagged particle diffusion in general (non-zero mean) non-reversible exclusion processes, in $d \geq 3$. The fundamental ideas related to the graded sector condition have their origin partly in [6].
- (6) For a list of applications of Theorem KV together with the strong and graded sector conditions, see the surveys [8], [5], and for a more recent application of the graded sector condition to the so-called *myopic self-avoiding walks* and *Brownian polymers*, see [3].

From abstract functional analytic considerations, it follows that the following two conditions jointly imply (2) and (3):

$$f \in \text{Ran}(S^{1/2}) \tag{5}$$

$$\sup_{\lambda > 0} \| S^{-1/2} G u_\lambda \| < \infty. \tag{6}$$

Checking conditions (2) and (3) (or (4), or (6)) in particular applications is typically not easy. In the applications to RWRE in [11], the conditions were checked by bare hand computations. In [12], respectively, [10], the sector condition, respectively, the graded sector condition were introduced and checked for the respective models.

3 Sector conditions

We first recall the (strong) sector condition and the graded sector condition. They are reformulated for the purpose of consistent notation throughout the present note. For other formulations, see the original papers [12], [10] and also the surveys [8], [5]. The following versions are closest in spirit to [8].

Theorem 2 (Strong sector condition). *With notations as before, if*

$$\| S^{-1/2} A S^{-1/2} \| < \infty,$$

then (6) follows, so for every function f for which (5) holds, the martingale approximation and CLT of Theorem KV applies automatically.

For the graded sector condition, assume that the Hilbert space $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$ is graded

$$\mathcal{H} = \overline{\oplus_{n=0}^{\infty} \mathcal{H}_n} \tag{7}$$

where \mathcal{H}_0 is the 1-dimensional subspace of constant functions. Since we work with functions f for which $\int_{\Omega} f \, d\pi = 0$, we exclude the subspace \mathcal{H}_0 from \mathcal{H} without changing the notation (and thus abusing it slightly).

Also, assume that the infinitesimal generator is consistent with the grading in the following sense:

$$S = \sum_{n=1}^{\infty} S_{n,n}, \quad S_{n,n} : \mathcal{H}_n \rightarrow \mathcal{H}_n, \quad S_{n,n}^* = S_{n,n}, \quad (8)$$

$$A = \sum_{n=1}^{\infty} \sum_{j=-r}^r A_{n+j,n}, \quad A_{n+j,n} : \mathcal{H}_n \rightarrow \mathcal{H}_{n+j}, \quad A_{n,m}^* = -A_{m,n} \quad (9)$$

for some fixed $r \in \mathbb{Z}^+$.

Note that $S_{n,n}$ and $A_{n+j,n}$ are not necessarily bounded operators; $\mathcal{C}_n \subseteq \mathcal{H}_n$ denotes a common core for them.

Theorem 3 (Graded sector condition). *Let the Hilbert space and the infinitesimal generator be graded in the sense specified above. If with some $\beta < 1$ and some $C < \infty$*

$$\left\| S_{n+j,n+j}^{-1/2} A_{n+j,n} S_{n,n}^{-1/2} \right\| \leq C n^{\beta}, \quad -r \leq j \leq r$$

holds, then (6) follows, so again, for any function f for which (5) holds, the martingale approximation and CLT of Theorem KV applies.

The statement remains valid even in the case $\beta = 1$ if C is small enough.

It is straightforward that Theorem 2 follows from the more sophisticated Theorem 3. Next we state a more general sector condition which implies and slightly improves the above Theorem 3. We call it *relaxed sector condition*. The existence of a grading is not assumed at this point.

Define *formally* for the moment

$$B_{\lambda} := (S + \lambda)^{-1/2} A (S + \lambda)^{-1/2}, \quad \lambda > 0, \quad (10)$$

$$B := B_0 = S^{-1/2} A S^{-1/2}. \quad (11)$$

A condition of the next theorem is that these formally defined objects make sense as closed and skew-self-adjoint operators.

Theorem 4 (Relaxed sector condition). *Assume that the following conditions hold.*

- (1) *The operators B_{λ} , $\lambda > 0$, and B , are defined on a common dense core denoted by $\mathcal{C} \subseteq \mathcal{H}$, and they are skew self-adjoint operators:*

$$B_{\lambda}^* = -B_{\lambda}, \quad B^* = -B. \quad (12)$$

(2) For any vector $f \in \mathcal{C}$

$$\lim_{\lambda \rightarrow 0} \| (B_\lambda - B)f \| = 0. \quad (13)$$

In this case, for any f for which (5) holds, the conclusions of Theorem KV apply.

Remarks. (1) Theorem 4 looks like an abstract statement. In fact, checking conditions (12) and (13) in particular cases is a very concrete task.

(2) The *(strong) sector condition*, cf. [12], is equivalent to the operator B in (11) being bounded. The *graded sector condition*, cf. [10], says that the operator B restricted to the ‘grade subspaces’ \mathcal{H}_n of the Hilbert space \mathcal{H} , is bounded with an increase restriction on the bound as $n \rightarrow \infty$. Our *relaxed sector condition* is less restrictive and seems to fit naturally to some applications.

(3) The core \mathcal{C} is not necessarily the same as \mathcal{C}_0 , the common core of G and G^* . Choosing properly the core \mathcal{C} is an important part of applying the theorem.

Next we show how a slightly stronger version of the *graded sector condition* follows from the *relaxed sector condition*. We assume that a grading consistent with the operators is given, as in (7)–(9).

Theorem 5 (Graded sector condition, stronger version). Assume that

$$\left\| S_{n+j,n+j}^{-1/2} A_{n+j,n} S_{n,n}^{-1/2} \right\| \leq c(n), \quad -r \leq j \leq r \quad (14)$$

where $c(n)$ is non-decreasing and

$$\sum_{n=0}^{\infty} \frac{1}{c(n)} = \infty. \quad (15)$$

In this case, for any function f for which (5) holds, the conclusions of Theorem KV apply.

4 Proof of sector condition theorems

4.1 Graded sector condition theorem follows from relaxed sector condition theorem

Proof of Theorem 5. We assume that (14) and (15) hold and we check the two conditions of Theorem 4. The core shall be the functions of finite grade, i.e.

$$\mathcal{C} = \oplus_{n=1}^{\infty} \mathcal{H}_n.$$

(Note that there is no closure of the orthogonal sum on the right hand side.)

From (14), B is clearly defined on \mathcal{C} . To see that B_λ is also defined on \mathcal{C} for $\lambda > 0$, note that

$$B_\lambda = S^{1/2}(\lambda + S)^{-1/2} B S^{1/2}(\lambda + S)^{-1/2}.$$

The operator $S^{1/2}(\lambda + S)^{-1/2}$ maps \mathcal{H}_n to \mathcal{H}_n , and from the spectral theorem to the self-adjoint operator S , it is obvious that

$$\| \lambda^{1/2}(\lambda + S)^{-1/2} \| \leq 1, \quad \lambda^{1/2}(\lambda + S)^{-1/2} \xrightarrow{\text{st.op.top.}} 0, \quad (16)$$

$$\| S^{1/2}(\lambda + S)^{-1/2} \| \leq 1, \quad S^{1/2}(\lambda + S)^{-1/2} \xrightarrow{\text{st.op.top.}} I, \quad (17)$$

hence, due to (17), B_λ for $\lambda > 0$ are also defined on \mathcal{C} . It is also straightforward that (13) follows from (17) and the fact that the operator $S^{1/2}(\lambda + S)^{-1/2}$ acts diagonally on the grading. Note that so far, the only assumptions used were that (14) holds for some finite values of $c(n)$, (15) was not used yet.

Direct calculation shows that B and B_λ are skew symmetric (antisymmetric). However, this is short of being skew self-adjoint (anti self-adjoint). We prove skew self-adjointness of B , the proof for skew self-adjointness of B_λ with $\lambda > 0$ is done similarly.

We refer to the "basic criterion for self-adjointness", see Theorem VIII.3 in [9]. According to this, we have to prove that

$$(\text{Ran}(B \pm I))^\perp = \{0\}.$$

We will prove $(\text{Ran}(B - I))^\perp = \{0\}$. (The proof for $B + I$ is identical.)

Let

$$x = (x_1, \dots) \in (\text{Ran}(B - I))^\perp$$

decomposed according to the grading and denote

$$x^n := (x_1, \dots, x_n, 0, 0, \dots) \in \mathcal{C}.$$

Then (using natural notation)

$$\begin{aligned} \langle x, x^n \rangle &= \langle x, Bx^n \rangle = \underbrace{\langle x^n, Bx^n \rangle}_0 + \langle x - x^n, Bx^n \rangle = \\ &= \sum_{i=1}^r \left\langle x_{n+i}, \sum_{j=i}^r S_{(n+i), (n+i)}^{-1/2} A_{(n+i), (n+i-j)} S_{(n+i-j), (nn+i-j)}^{-1/2} x_{n+i-j} \right\rangle. \end{aligned}$$

We estimate the right-hand side, by using (14). We get that

$$\begin{aligned} |\langle x, x^n \rangle| &\leq \sum_{i=1}^r \sum_{j=i}^r c(n+i-j) \frac{\|x_{n+i}\|^2 + \|x_{n+i-j}\|^2}{2} \\ &\leq rc(n) \sum_{k=-r+1}^r \|x_{n+k}\|^2. \end{aligned}$$

If n is large enough, say $n \geq n_0$, then

$$\langle x, x^n \rangle \geq \frac{1}{2} \|x\|^2,$$

and so

$$\begin{aligned} \frac{1}{2} \sum_{n=n_0}^{\infty} \frac{1}{c(n)} \|x\|^2 &\leq \sum_{n=n_0}^{\infty} \frac{1}{c(n)} \langle x, x^n \rangle \\ &\leq r \sum_{n=n_0}^{\infty} \sum_{j=-r+1}^r \|x_{n+j}\|^2 \leq 2r^2 \sum_{n_0-r+1}^{\infty} \|x_n\|^2. \end{aligned}$$

This, along with (15), imply $x = 0$, as requested. □

4.2 Proof of the relaxed sector condition theorem

Proof of Theorem 4. We assume that the operators B_λ for $\lambda \geq 0$ defined in (10) and (11) are skew self-adjoint. (Actually, slightly less is needed. As before, by (17) and the existence of a common core for B and B_λ ($\lambda > 0$) imply that it is enough to prove that B is skew self-adjoint, and it follows automatically that $B_\lambda, \lambda > 0$ are skew self-adjoint too.)

The following operators are well-defined and *bounded*:

$$K_\lambda := (I - B_\lambda)^{-1}, \quad \|K_\lambda\| = 1.$$

It is straightforward that we can write

$$R_\lambda = (\lambda + S)^{-1/2} K_\lambda (\lambda + S)^{-1/2}. \quad (18)$$

Proposition 1. *The following two conditions imply (2) and (3):*

(1) *The sequence $K_\lambda \in \mathcal{B}(\mathcal{H})$ converges in the strong operator topology:*

$$K_\lambda \xrightarrow{\text{st.op.top.}} K \in \mathcal{B}(\mathcal{H}) \quad \text{as } \lambda \rightarrow 0. \quad (19)$$

(2) *$f \in \text{Ran}(S^{1/2})$, or equivalently,*

$$\|S^{-1/2}f\|^2 := \lim_{\lambda \rightarrow 0} \|(\lambda + S)^{-1/2}f\|^2 = \lim_{\lambda \rightarrow 0} (f, (\lambda + S)^{-1}f) < \infty. \quad (20)$$

Proof. We write

$$g := S^{-1/2}f \in \mathcal{H}.$$

By condition (20), $g \in \mathcal{H}$, and using (18), we get

$$\lambda^{1/2}u_\lambda = \lambda^{1/2}(\lambda + S)^{-1/2}K_\lambda(\lambda + S)^{-1/2}S^{1/2}g, \quad (21)$$

$$S^{1/2}u_\lambda = S^{1/2}(\lambda + S)^{-1/2}K_\lambda(\lambda + S)^{-1/2}S^{1/2}g. \quad (22)$$

From (19), (21), (22), (16) and (17), we readily get (2) and (3) with

$$v = Kg.$$

□

In the next proposition, we formulate a practically usable sufficient condition for (19) to hold. This is reminiscent of Theorem VIII.25(a) from [9]:

Proposition 2. *Let B_n , $n \in \mathbb{N}$, and $B = B_\infty$ be densely defined closed operators over the Hilbert space \mathcal{H} . Assume that*

(1) *Some (fixed) $\mu \in \mathbb{C}$ is in the resolvent set of all operators B_n , $n \leq \infty$, and*

$$\sup_{n \leq \infty} \|(\mu I - B_n)^{-1}\| < \infty.$$

(2) *There is a (dense) subspace $\mathcal{C} \subseteq \mathcal{H}$ which is a core for B_∞ and $\mathcal{C} \subseteq \text{Dom}(B_n)$, $n < \infty$, such that for all $h \in \mathcal{C}$:*

$$\lim_{n \rightarrow \infty} \|B_n h - B h\| = 0.$$

Then

$$(\mu I - B_n)^{-1} \xrightarrow{\text{st.op.top.}} (\mu I - B)^{-1}. \quad (23)$$

Proof. Since \mathcal{C} is a core for the densely defined closed operator B and μ is in the resolvent set of B , the subspace

$$\tilde{\mathcal{C}} := \{\tilde{h} = (\mu I - B)h : h \in \mathcal{C}\}$$

is dense in \mathcal{H} . Thus, for any \tilde{h} from this dense subspace, we have

$$\{(\mu I - B_n)^{-1} - (\mu I - B)^{-1}\}\tilde{h} = (\mu I - B_n)^{-1}(B_n h - B h) \rightarrow 0.$$

This implies (23). □

Putting Propositions 1 and 2 together, we obtain Theorem 4. □

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